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ASYMPTOTIC ANALYSIS OF CONVECTIVE DIFFUSION IN PROBLEMS WITH A DISCONTINUITY IN THE CATALYTIC PROPERTIES OF THE SURFACE AROUND WHICH THE FLOW TAKES PLACE*

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The stationary concentration distribution in the flow round bodies with a discontinuity in their surface catalytic properties is investigated. An asymptotic analysis of this problem is carried out on the basis of the Navier-Stokes equations when $\mathrm{Re}\to\infty$ in the neighbourhood of the point of discontinuity in the catalytic properties, and the corresponding boundary value problems for the leading terms in the expansion of the required functions are formulated. Two spatial problems are solved in which account is taken of the transverse diffusion during the circumfluence of a planar surface with a rectangular insert endowed with different catalytic properties. Cases are considered when the diffusion flux of recombining particles changes in a stepwise manner on-passing over the surface of the insert and when the main surface is non-catalytic but the insert is ideally catalytic.

1. Let us consider the stationary flow of a bindary mixture of a chemically reacting compressible gas along the surface of a disc and let this disc have a discontinuity in its surface catalytic properties at a distance $x_0 = O(1)$ from the origin. We assume that $\varepsilon = \mathrm{Re}^{-1/2}\to 0$, $\mathrm{Re} = \rho_\infty V_\infty x_0/\mu_\infty$ (quantities with the subscript ∞ correspond to values of the parameters in the approach stream). In a rectangular system of Cartesian coordinates x, y (see /1/, for example), the Navier-Stokes equations in the dimensionless variables

$$x^* = \frac{x-x_0}{x_0},\quad y^* = \frac{y}{x_0},\quad \rho^* = \frac{\rho}{\rho_\infty},\quad \mathbf{v}^* = \frac{\mathbf{v}}{V_\infty},\quad k^* = \frac{k}{V_\infty}$$

$$T^* = \frac{T}{T_\infty},\quad T_\infty = \frac{V_\infty^2}{c_{p\infty}},\quad p^* = \frac{p}{\rho_\infty V_\infty^2},\quad h^* = \frac{h}{c_{p\infty}T_\infty},$$

$$w^{\cdot *} = \frac{w^{\cdot} x_0}{\rho_\infty V_\infty}$$

take the form (we shall omit the asterisks above the dimensionless quantities)

$$\nabla\cdot\rho\mathbf{v} = 0,\ \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \varepsilon^2\nabla\cdot\boldsymbol{\tau} \tag{1.1}$$
$$\rho\mathbf{v}\cdot\nabla c = \varepsilon^2\,\nabla(\mu\,\mathrm{Sc}^{-1}\nabla c) + w^{\cdot}$$

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$$\rho c_p \mathbf{v}\cdot\nabla T = \mathbf{v}\cdot\nabla p - (h_a - h_m)w^{\cdot} + \varepsilon^2 [\tau : \mathbf{e} + \nabla (\mu c_p \sigma^{-1}\nabla T) + (c_{pa} - c_{pm})\mu\, \mathrm{Sc}^{-1}\, \nabla c\cdot\nabla T]$$

$$p = \frac{\rho}{m} RT, \quad R = \frac{R_A}{c_{p\infty}}, \quad \frac{1}{m} = \frac{c}{m_a} + \frac{c_m}{m_m}, \quad \nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

$$\mathbf{e} = \| e_{\alpha\beta} \|, \quad \tau = \| \tau_{\alpha\beta} \|, \quad \tau_{\alpha\beta} = -{}^2/_3\, \delta_{\alpha\beta}\nabla\cdot\mathbf{v} + 2\mu e_{\alpha\beta}$$

Quantities with the subscripts a and m refer to atoms and molecules respectively, $\mathbf{v} = (u, v)$; ρ, p, T, and h are the velocity, density, pressure, temperature and enthalpy, c is the mass concentration of the atoms, μ, c_p, Sc and σ are the coefficient of viscosity, the specific heat capacity and the Schmidt and Prandtl numbers, $w^{\cdot}$ and R_A are the chemical source term and the universal gas constant and $\mathbf{e}$ is the deformation rate tensor.

We shall specify the boundary conditions on the body in the following form (we assume that the dimensionless rate constant of the heterogeneous reaction is of the order of $\varepsilon^{\alpha_i} k_i$, $k_i = O(1)$):

$$\mathbf{v} = 0, \ T = T_w(x), \ \mu\, \mathrm{Sc}^{-1}\, \partial c/\partial y = \varepsilon^{\alpha_i - 2}\rho k_i c, \ y = 0 \tag{1.2}$$
$$k = k_1, \ x < 0; \ k = k_2, \ x \geqslant 0$$

Let us assume that the external flow (unperturbed by the discontinuity in the catalytic properties) is described by the boundary-layer equations. Then,

$$A \equiv \partial T/\partial\eta = O(1), \ U \equiv \partial u/\partial\eta = O(1) \tag{1.3}$$

where $\eta = y/\varepsilon$ is the boundary layer variable.

We shall further assume that the quantities $c_{pa}(T)$, $h_a(T)$, $c_{pm}(T)$, $h_m(T)$, Sc, μ, σ, $w^{\cdot}$ are of the order of unity and that the order of magnitude of the atomic concentration c when $x = 0$ is determined from the solution of the external problem. To be specific, let us postulate that, in the outer unperturbed boundary layer, the convective thermal flux terms, due to thermal conduction and diffusion, are of the same order of magnitude:

$$\partial T/\partial\eta \sim \partial c/\partial\eta \sim O(1) \tag{1.4}$$

It then follows from (1.2) and (1.4) that, when $\eta = 0$, $x = 0$, that is, as the point of discontinuity is approached from the left,

$$c = O(\varepsilon^{1-\alpha_1}), \ \alpha_1 \leqslant 1 \tag{1.5}$$

Let us now first consider the case when $\alpha_1 \leqslant \alpha_2$ (a stepwise transition to a surface with a smaller or equal catalytic activity) and introduce the new variables and functions (the quantities with zero subscript are found from the solution of the external parabolic problem when $x = 0-$, $\eta = 0$):

$$x = \varepsilon^{3/2}x_1, \ y = \varepsilon^{3/2}y_1, \ \mathbf{v} = \varepsilon^{1/2}\mathbf{v}_1 + \ldots, \ \tau = \varepsilon^{-1}\tau_1 + \ldots, \tag{1.6}$$
$$T = T_{w0} + \varepsilon^{1/2}T_1 + \ldots, \ c = c_0 + \varepsilon^{1/2}c_1 + \ldots,$$
$$p = p_0 + \varepsilon p_1 + \ldots, \ \rho = \rho_0 + \ldots, \ \mu = \mu_0 + \ldots,$$
$$\mathrm{Sc} = \mathrm{Sc}_0 + \ldots, \ \sigma = \sigma_0 + \ldots, \ c_p = c_{p0} + \ldots$$

where the string of dots denotes terms with a higher order of smallness compared with the first term. The form of the expansions for $\mathbf{v}$, T and c is determined from the conditions for matching with the external solution of (1.3), (1.4) and (1.5) and by the boundary conditions when $y = 0$. The form of the expansions for x and y is determined from the condition that, in the neighbourhood of the point of discontinuity in the surface catalytic properties, the linear scales for the development of diffusion are of the same order of magnitude in the longitudinal (along the disc) and transerve directions and the condition that the convective and diffusion terms in the diffusion equation are of the same order of magnitude.

Let us now consider the boundary conditions for the concentration c on the surface $y_1 = 0$ and put $c_0(\varepsilon) = c_0'\varepsilon^{1-\alpha_1}$, $c_0' = O(1)$ (see (1.5)). By substituting the expansion for c (1.6) into (1.2), we obtain, as $\varepsilon \to 0$,

$$\partial c_1/\partial y_1 = K_1 (c_0' + c_1\varepsilon^{\alpha_1 - 1/2}), \ x_1 < 0 \tag{1.7}$$
$$\partial c_1/\partial y_1 = K_2 (c_0'\varepsilon^{\alpha_2 - \alpha_1} + c_1\varepsilon^{\alpha_2 - 1/2}), \ x_1 \geqslant 0$$
$$K_i = \rho_0 k_i\, \mathrm{Sc}_0/\mu_0$$

Depending on the values of the quantities α_1 and α_2 (we have already noted that $\alpha_1 \leqslant \alpha_2$), four forms of conditions (1.7) are possible;

$$\partial c_1/\partial y_1 = K_1 c_0', \ x_1 < 0 \tag{1.8}$$
$$\partial c_1/\partial y_1 = K_2 c_0' \ (\alpha_2 = \alpha_1) \ \partial c_1/\partial y_1 = 0 \ (\alpha_1 < \alpha_2), \ x_1 \geqslant 0$$

when $\alpha_1 = 1/2$

$$\partial c_1/\partial y_1 = K_1 (c_0' + c_1),\ x_1 < 0 \qquad (1.9)$$
$$\partial c_1/\partial y_1 = K_2 (c_0' + c_1)\ (\alpha_2 = \alpha_1),\ \ \partial c_1/\partial y_1 = 0\ (\alpha_1 < \alpha_2),\ x_1 \geqslant 0$$

when $\alpha_1 < 1/2 \leqslant \alpha_2$

$$c_1 = 0,\ x_1 < 0 \qquad (1.10)$$
$$\partial c_1/\partial y_1 = K_2 c_1\ (\alpha_2 = 1/2),\ \partial c_1/\partial y_1 = 0\ (\alpha_2 > 1/2),\ x_1 \geqslant 0$$

when $\alpha_1 \leqslant \alpha_2 < 1/2$

$$c_1 = 0,\ x_1 < 0;\ c_1 = 0,\ x_1 \geqslant 0 \qquad (1.11)$$

Substituting (1.6) into (1.1), we obtain, as $(\varepsilon \to 0)$

$$\rho_0 \nabla_1 \cdot \mathbf{v}_1 = 0,\ \rho_0 \mathbf{v}_1 \cdot \nabla_1 \mathbf{v}_1 = -\nabla_1 p_1 + \mu_0 \nabla_1^2 \mathbf{v}_1 \qquad (1.12)$$
$$\rho_0 \mathbf{v}_1 \cdot \nabla_1 c_1 = \mu_0 \mathrm{Sc}_0^{-1} \nabla_1^2 c_1$$
$$\rho_0 c_{p0} \mathbf{v}_1 \cdot \nabla_1 T_1 = \mu_0 \sigma_0^{-1} c_{p0} \nabla_1^2 T_1$$
$$\nabla_1 \equiv (\partial/\partial x_1,\ \partial/\partial y_2)$$

for the first terms in the expansion.

The boundary conditions on the surface for the system of Eqs.(1.12) will have the form

$$\mathbf{v}_1 = 0,\ T_1 = 0,\ y_1 = 0 \qquad (1.13)$$

while, in the case of c_1, the conditions are (1.8)-(1.11).

The boundary conditions at infinity are determined by the conditions for matching with the solution in the boundary layer:

$$u_1 = U y_1,\ T_1 = A y_1,\ c_1 = K_1 c_0' y_1,\ p_1 \to 0\ x_1 \to -\infty \qquad (1.14)$$
$$\partial u_1/\partial y_1 = U,\ \partial T_1/\partial y_1 = A,\ \partial c_1/\partial y_1 = K_1 c_0',\ \ p_1 \to 0,\ y_1 \to \infty$$

As $x \to \infty$, the boundary conditions are determined by the approach to the boundary-layer solution:

$$\partial u^2/\partial x_1^2 = 0,\ \partial^2 T/\partial x_1^2 = 0,\ \ \partial^2 c_1/\partial x_1^2 = 0,\ \ p_1 = 0,\ \ x_1 \to \infty \qquad (1.15)$$

The system of Eqs.(1.12) (with the exception of the fourth equation) is identical with the system for an incompressible viscous fluid with constant properties and admit of the obvious solutions of the dynamic and thermal problems

$$u_1 = U y_1,\ v_1 = 0,\ p_1 = 0,\ T_1 = A y_1$$

The diffusion problem is separated from the dynamic and thermal problems and reduces to the solution of the equation

$$U y_1 \partial c_1/\partial x_1 = \mu_0\, \mathrm{Sc}_0^{-1} \nabla_1^2 c_1 \qquad (1.16)$$

and, in the case of boundary condition (1.11) has the trivial solution $c_1 = K_1 c_0' y_1$. In order to take account of the discontinuity in the boundary conditions, the following terms in the expansion are necessary.

Let us now consider the case when $\alpha_2 < \alpha_1$, that is, when there is a stepwise transition towards the surface with the greater catalytic activity. For simplicity, let us assume that $\alpha_1 = 1$. Then, $c = c_0 = O(1)$ when $\eta = 0,\ x = 0-$.

When $1/2 < \alpha_2 < 1$, the solution for all of the functions (with the exception of c) will be sought in the form of (1.6), while the solution in the case of the concentration will be sought in the form

$$c = c_0 + \varepsilon^{\alpha_2 - 1/2} c_1 + \ldots \qquad (1.17)$$

By subsituting the expansions (1.6), (1.17) into (1.1) and (1.2) we obtain the system (1.12) with boundary conditions (1.13)-(1.15) for the first terms of the expansion in the case of $\mathbf{v}_1, T_1$, and p_1. The boundary conditions for the concentration will have the form

$$\partial c_1/\partial y_1 = 0,\ x_1 < 0;\ \ \partial c_1 \partial y_1 = K_2 c_0,\ \ x_1 \geqslant 0,\ y_1 = 0 \qquad (1.18)$$
$$c_1 = 0,\ y_1 \to \infty;\ c_1 = 0,\ x_1 \to -\infty;\ \partial^2 c_1/\partial x_1^2 = 0,\ x_1 \to \infty$$

It is obvious that, in this case, the solution of the diffusion problem reduces to the solution of Eq.(1.16) with the boundary conditions (1.18). We note that Eq.(1.16) with boundary conditions (1.8)-(1.10), (1.18) and (1.14), (1.15) has been solved in a number of papers both analytically /2-5/ and numerically /6, 7/.

When $\alpha_2 < 1/2$, it follows from (1.2) that, in this case, the change in the concentration in the neighbourhood of the point of discontinuity will be a quantity of the order of unity.

We shall seek the solution for $x, y, \mathbf{v}, \boldsymbol{\tau}, T$ and p in the form of (1.6) while, for the remaining functions, we shall have

$$c = c_1 + \ldots, \rho = \rho_1 + \ldots, \mathrm{Sc} = \mathrm{Sc}_1 + \ldots, \sigma = \sigma_1 + \ldots, \quad \mu = \mu_1 + \ldots, c_p = c_{p1} + \ldots \tag{1.19}$$

Substituting expansions (1.6), (1.19) into (1.1), we obtain, as $\varepsilon \to 0$, the system

$$\begin{gathered}\nabla_1 \cdot \rho_1 \mathbf{v}_1 = 0, \ \rho_1 \mathbf{v}_1 \cdot \nabla_1 \mathbf{v}_1 = -\nabla_1 p_1 + \nabla_1 \boldsymbol{\tau}_1 \\ \rho_1 \mathbf{v}_1 \cdot \nabla c_1 = \nabla_1 (\mu_1 \mathrm{Sc}_1^{-1} \nabla_1 c_1) \\ \rho_1 c_{p1} \mathbf{v}_1 \cdot \nabla_1 T_1 = \nabla_1 (\mu_1 c_{p1} \sigma_1^{-1} \nabla_1 T_1) + (c_{pa} - c_{pm}) \mu_1 \mathrm{Sc}_1^{-1} \nabla_1 c_1 \cdot \nabla_1 T_1 \\ \rho_1 / m = p_0 / (T_{w0} R) = \mathrm{const}\end{gathered} \tag{1.20}$$

for the principal and first terms.

The boundary conditions for $\mathbf{v}_1$, p_1 and T_1 are the earlier boundary conditions ((1.14) and (1.15)) while, for the concentration, they will be

$$\begin{gathered}\partial c_1 / \partial y_1 = 0, \ x_1 < 0; \ \partial c_1 / \partial y_1 = \rho_1 k_2 \mathrm{Sc}_1 c_1 / \mu_1 \ (\alpha_2 = 1/2), \ c_1 = 0 \\ (\alpha_2 < 1/2), \ x_1 \geqslant 0, \ y_1 = 0 \\ c_1 = c_0, \ x_1 \to -\infty; \ c_1 = c_0, \ y_1 \to \infty; \\ \partial^2 c_1 / \partial x_1^2 = 0, \ x_1 \to \infty\end{gathered} \tag{1.21}$$

In the case when $\alpha_2 < 1/2$, the zone of relaxation to equilibrium was neglected.

Let us now consider the circumfluence of a smooth convex planar profile in the above-mentioned formulation and introduce the following natural coordinate system. The x-axis (the length of an arc) is directed along the surface of the body and the y-axis along the normal to it. It is obvious that, on account of the local nature of the analysis which has been carried out, the problem on the discontinuity in the catalytic behaviour reduces in this case to that which has just been considered, that is, (1.16) or (1.20) with the corresponding boundary conditions. The quantities with the zero subscript are equal to their values obtained from the solution of the external parabolic problem on the surface of the body when $x = x_0 - 0$ in the natural coordinate system.

Hence, on passing from the surface for which the catalytic recombination constant is $\varepsilon^{\alpha_1} k_1$ to a surface where this constant is equal to $\varepsilon^{\alpha_2} k_2$, when $\alpha_1 \leqslant \alpha_2$, the flow in the neighbourhood of the discontinuity is described by Eq.(1.16) with the boundary conditions (1.8)-(1.11), (1.14), (1.15). If $\alpha_1 = 1 > \alpha_2$, then, when $1/2 < \alpha_2 < 1$, the flow is also described by (1.16) with the boundary conditions (1.18). When $\alpha_2 < 1/2$, the flow is described by the boundary value problem (1.20), (1.14), (1.15) and (1.21). The size of the perturbed domain is a quantity of the order of $\varepsilon^{3/2} = \mathrm{Re}^{-3/4}$.

2. Let us now obtain a solution of the diffusion problem when, on the planar surface, there is a rectangular insert with other catalytic properties, the sides of which are parallel and perpendicular to the approach stream and are described by the equations $z = 0$ and $x = 0$ respectively. On passing onto the surface of the insert, the concentration flux changes in a stepwise manner. We start out from the diffusion equation written in the form (cf. (1.16))

$$y \partial c / \partial x = \partial^2 c / \partial y^2 + \partial^2 c / \partial z^2 \tag{2.1}$$

Let us specify boundary conditions of the type (1.8) when $y = 0$. Then, without loss of generality, it is sufficient to consider the following conditions:

$$\begin{gathered}c = 0, \ x = 0; \ c = 0 \ \ y \to \infty; \ \partial c / \partial y = -1, z < 0, \ \partial c / \partial y = 1, \\ z > 0, \ y = 0, \ \partial^2 c / \partial z^2 = 0, \ z \to \pm\infty\end{gathered} \tag{2.2}$$

A similar problem has been solved numerically in /7/. The term $\partial^2 c / \partial x^2$, omitted from (2.1), is negligibly small when $x \geqslant 1$ /7/.

Let us carry out a Laplace transform with respect to x /8/

$$f(p, y, z) = \int_0^\infty \exp(-px)\, c(x, y, z)\, dx$$

Using the method of separation of variables, we get

$$f = \frac{1}{\pi p} \int_{-\infty}^{+\infty} \varphi(y, \lambda) \frac{\sin \lambda z}{\lambda} d\lambda, \quad \varphi = \frac{\mathrm{Ai}\,(p^{-2/3}(yp + \lambda^2))}{p^{1/3} \mathrm{Ai}'\,(\lambda^2 p^{-2/3})} \tag{2.3}$$

where $\mathrm{Ai}(s)$ and $\mathrm{Ai}'(s)$ are the Airy function of the first kind and its derivative. A cut is made in the p-plane along the negative real axis and the branches of the functions $(\alpha+i\beta)^r$ are fixed by the condition $(\alpha+i\beta)^r \to \alpha^r$ as $\alpha \to +\infty$. Hence, φ decays exponentially as $y \to \infty$ /9/.

We note that $c(x, y, z) = -c(x, y, -z)$ and we shall therefore subsequently assume that $z > 0$. By closing the path of integration in (2.3) with an arc of a circle $|\lambda| = R$ in the upper half of the λ-plane and using Jordan's lemma /8/ (the zeros of the function $\mathrm{Ai}'(\lambda^2 p^{-2/3})$ where p is fixed and $p \neq 0$ are simple), we get

$$f = \frac{1}{p^{4/3}}\left[\frac{\mathrm{Ai}(yp^{1/3})}{\mathrm{Ai}'(0)} - \sum_{n=0}^{\infty}\frac{\mathrm{Ai}(yp^{1/3}-a_n')}{\mathrm{Ai}''(-a_n')\,a_n'}\exp(-a_n'^{1/2}p^{1/3}z)\right] \tag{2.4}$$

where $-a_n'$ $(n = 0, 1, 2, \ldots)$ are the zeros of the function $\mathrm{Ai}'(s)$. It follows from (2.4) that conditions (2.2) are satisfied as $z \to \pm\infty$.

Next, let us seek a solution when $y = 0$. Then, (2.4) takes the form

$$f(p, 0, z) = \frac{A + \Sigma(z)}{p^{4/3}}; \quad \Sigma(z) = \sum_{n=0}^{\infty}\frac{\exp(-a_n'^{1/2}p^{1/3}z)}{a_n'^2}, \quad A = \frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)} \tag{2.5}$$

It follows from (2.5) that $f = 0$ when $z = 0$ since the equality $A + \Sigma(0) = 0$ is valid. In order to prove this equality, it is sufficient to consider (in the sense of the principal value) the integral

$$\int_{-\infty}^{+\infty}\frac{\mathrm{Ai}(\lambda^2)}{\lambda\,\mathrm{Ai}'(\lambda^2)}\,d\lambda = 0$$

and to close the integration path in the upper half of the λ-plane.

By applying the inverse Laplace transformation formula /8/ to f, we obtain

$$c = -\frac{3^{2/3}x^{1/3}}{\Gamma(2/3)} + \frac{1}{2\pi i}\int_{-i\infty+b}^{i\infty+b}\sum_{n=0}^{\infty}\frac{\exp(-a_n'^{1/2}p^{1/3}z + px)}{a_n'^2 p^{4/3}}\,dp, \quad b > 0 \tag{2.6}$$

from (2.5) where use has been made of the fact that $A = \mathrm{Ai}(0)/\mathrm{Ai}'(0) = -\Gamma(1/3)/(3^{1/3}\Gamma(2/3))$ /9/. The first term in (2.6) corresponds exactly to the parabolic solution of system (2.1), (2.2) (when $\partial^2 c/\partial z^2$ is neglected) when $z > 0$. The second terms describes the effect of transverse diffusion (directed along the z-axis). It follows from (2.6) that $cx^{-1/3}$ is the function $zx^{-1/3}$, that is, the size of the domain influenced by transverse diffusion increases as $z \sim x^{1/3}$.

On account of the rather complex expression (2.6), we shall confine ourselves to obtaining the main terms of the asymptotic expansion for c at small and large values of z.

By means of the Euler-McLaurin theorem /10/, we obtain, allowing for the asymptotic nature of the behaviour of the zeros a_n' as $n \to \infty$, /9/ that

$$\Sigma(1) = -A + 2\pi^{-1}p^{1/3}\ln p^{1/3} + O(p^{1/3}) \tag{2.7}$$

After making the substitution $q = pz^{1/3}$ in the integral (2.6), we substitute (2.7) into (2.6) and carry out a term by term integration. Then,

$$C \equiv cx^{-1/3} = 2\pi^{-1}Z\ln Z + O(Z), \quad Z = zx^{-1/3} \tag{2.8}$$

It follows from (2.8) that, in the neighbourhood of the line of discontinuity $Z = 0$, the function C is continuous and $\partial C/\partial Z$ has a logarithmic singularity.

In the case when $Z \gg 1$, we proceed in the following manner. By applying criterion B /11/, it can be proved that term by term integration when $z > 0$ in (2.6) is valid. By making the substitution $q' = pxa_n'^{-3/4}Z^{-3/2}$ in (2.6) and applying the method of steepest descent /12/, we obtain

$$C = -\frac{3^{1/3}}{\Gamma(2/3)} + \frac{3^{7/4}}{2\pi^{1/2}a_0'^{23/8}Z^{5/4}}\exp\left(-\frac{2a_0'^{3/4}Z^{3/2}}{3\sqrt{3}}\right)\left(1 + O\,\frac{1}{Z^{3/2}}\right) \tag{2.9}$$

where $-a_0'$ is the value of the first zero (the minimum in absolute magnitude) of the function $\mathrm{Ai}'(s)$, $a_0' = 1.01879$. In (2.9), use has been made of the property that a_n' increases monotonically as n increases.

The effect of transverse diffusion therefore falls off exponentially when $Z \gg 1$.

The dependence $C(Z)$ is shown in Fig.1 where the solid line represents the numerical solution /7/ of problem (2.1), (2.2) obtained by the method of alternating directions while the broken line was obtained using formula (2.9). The dot-dash line corresponds to the boundary-layer soltuion $C = -3^{2/3}/\Gamma(^2/_3) = -1.536$. It is seen that the difference between the numerical solution and the asymptotic solution (Eq.(2.9)) is less than 5% when $Z \geqslant 2.5$ and that the asymptotic solution merges with the boundary-layer solution (a difference of less than 2%) when $Z \geqslant 3.5$

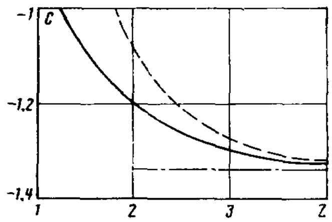

Fig.1

3. Let us now consider the spatial diffusion problem when the main surface is non-catalytic and the rectangular insert is ideally catalytic. The solution is assumed to be continuous and bounded. We note that this problem also arises in the theory of a film anemometer /13/.

Let us introduce the function $\theta = 1 - c$. Then, the problem takes the form

$$y\partial\theta/\partial x = \partial^2\theta/\partial y^2 + \partial^2\theta/\partial z^2 \tag{3.1}$$

$$\theta = 0,\ x = 0;\ \partial\theta/\partial y = 0,\ z < 0;\quad \theta = 1,\ z > 0,\ y = 0,\ x > 0$$

$$\theta = 1,\ y \to \infty;\ \theta \to 0,\ z \to -\infty$$

We replace the condition that $\theta = 1$ when $z > 0$ by

$$\theta(x, 0, z) = \psi,\ z > 0 \tag{3.2}$$

$$\psi = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \exp(-a_0 p^{1/3} z + px)\frac{dp}{p}$$

The p-plane is cut in the same way as in Sect.2 and a_0 is a small positive quantity introduced in order to guarantee the existence of the Fourier transform of θ and it will be allowed to tend to zero when this becomes convenient since (when $z > 0$)

$$\lim_{a_0 \to 0} \psi(a_0, x, z) = 1,\ \lim_{x \to 0} \psi(a_0, x, z) = 0$$

Let us now apply a Laplace transform with respect to x to the function θ. Then, from (3.1) and (3.2), we get

$$ypf = \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 \tag{3.3}$$

$$\partial f/\partial y = 0\ z < 0,\ f = \exp(-a_0 p^{1/3} z)/pz > 0,\ y = 0 \tag{3.4}$$

$$f \to 0,\ y \to \infty;\ f \to 0,\ z \to -\infty \tag{3.5}$$

Let us further assume that the last condition in (3.5) can be strengthened:

$$f = O(\exp(b_0 p^{1/3} z)),\ z \to -\infty \tag{3.6}$$

for all y and a certain real number $b_0 > 0$. The validity of this hypothesis will be confirmed below.

Problem (3.3)-(3.6) is solved by the Wiener-Hopf method /14/. By carrying out a Fourier transformation with respect to z, we get from (3.3), after the condition at infinity has been satisfied, that

$$\Phi = \Phi_+ + \Phi_- = H(p, k)\mathrm{Ai}(p^{-2/3}(yp + k^2)) \tag{3.7}$$

$$\Phi_+ = \int_{-\infty}^{0} fe^{-ikz}\,dz,\quad \Phi_- = \int_{0}^{\infty} fe^{-ikz}\,dz$$

where $\mathrm{Ai}(s)$ is the Airy function of the first kind.

From the boundary conditions when $y = 0$ and the requirement that the solution should be bounded, it can be deduced that the function Φ_- is analytic in the lower half-plane $(\mathrm{Im}(k) < a_0\,\mathrm{Im}(ip^{1/3}))$ and that Φ_+ is analytic in the upper half-plane $\mathrm{Im}(k) > -b_0\,\mathrm{Im}(ip^{1/3})$. The value of p is fixed and $p \neq 0$. When $y = 0$, we obtain

$$\Phi = \Phi_+ - \frac{i}{p(k - ia_0 p^{1/3})} = H\,\mathrm{Ai}(s) \tag{3.8}$$

$$\frac{\partial\Phi}{\partial y} = \frac{\partial\Phi_-}{\partial y} = Hp^{1/3}\mathrm{Ai}'(s),\quad s = k^2 p^{-2/3}$$

from (3.4), (3.7).

By eliminating H from (3.8), we find

$$\Phi_+ + \frac{1}{F}\frac{\partial\Phi_-}{\partial y} - \frac{i}{p(k - ia_0p^{1/3})} = 0, \quad F = -p^{1/3}\frac{\mathrm{Ai}'(s)}{\mathrm{Ai}(s)} \tag{3.9}$$

We shall use the Wiener-Hopf method to solve Eq.(3.9). Since the zeros of $\mathrm{Ai}(s)$ and $\mathrm{Ai}'(s)$ are located in the k-plane on the stright line $k = ip^{1/3}t$ (t is a real parameter) then, by choosing a_0 and b_0 to be smaller than the square root of the absolute value of the first zero of $\mathrm{Ai}'(s)$, we obtain that each term is definite and different in the band $-b_0 \,\mathrm{Im}\,(ip^{1/3}) < \mathrm{Im}\,(k) < a_0 \,\mathrm{Im}\,(ip^{1/3})$. We next find $\partial\Phi_-/\partial y$. For this purpose, let us represent the left-hand side of (3.9) as the sum of two functions which are analytic when $\mathrm{Im}\,(k) > -b_0\,\mathrm{Im}\,(ip^{1/3})$ and $\mathrm{Im}\,(k) < a_0\,\mathrm{Im}\,(ip^{1/3})$ respectively.

Let $F = F_+(k, p)F_-(k, p)$. The function F_+ (F_-) is analytic and does not have any zeros in the upper (lower) k half-plane (the expressions for F_+ and F_- (the factorization of F) will be obtained below). Eq.(3.9) can then be written in the form

$$\Phi_+F_-(k, p) - \frac{i\,(F_+(k, p) - F_+(ia_0p^{1/3}, p))}{p\,(k - ia_0p^{1/3})} = \tag{3.10}$$
$$- \frac{1}{F_-(k, p)}\frac{\partial\Phi_-}{\partial y} + \frac{iF_+(ia_0p^{1/3}, p)}{p\,(k - ia_0p^{1/3})}$$

Let us assume that $\partial c/\partial y$ has a singularity which can be integrated with respect to z as $z \to 0+$. Then, $\partial\Phi_-/\partial y \to 0$ as $|k| \to \infty$. According to the principle of analytic continuity, there exists an integral function $I(k, p)$ which is identical to each of the parts of Eq.(3.10) where this part is defined. It tends to zero for each fixed $p \neq 0$ as $k \to \infty$ since, as we shall show later, $|F_\pm| = O(|k|^{1/2})$. According to Liouville's theorem, $I(k, p) \equiv 0$ and therefore

$$\frac{\partial\Phi_-}{\partial y} = \frac{iF_+(ia_0p^{1/3}, p)\,F_-(k, p)}{p\,(k - ia_0p^{1/3})} \tag{3.11}$$

Let us now factorize the function F. Let $-a_n$ $(n = 1, 2, \ldots)$ and $-a_n'$ $(n = 0, 1, \ldots)$ be the zeros of the functions $\mathrm{Ai}(s)$ and $\mathrm{Ai}'(s)$. It can be proved that

$$\frac{\mathrm{Ai}'(k^2)}{\mathrm{Ai}(k^2)} = -\left(\frac{2}{3\pi}\right)^{1/3} e^{\gamma/3}(k^2 + a_0')\,\Pi_-(k)\,\Pi_+(k) \tag{3.12}$$
$$\Pi_\pm \equiv \Pi_\pm(k) = \prod_{n=1}^{\infty} \frac{k \pm ia_n'^{1/2}}{k \pm ia_n^{1/2}}\, e^{-1/(6n)}$$

where γ is Euler's constant.

It follows from (3.9) and (3.12) that

$$F_\pm = (23^{-1}\pi^{-1}e^{\gamma}p^{-1})^{1/6}(k \pm ia_0'^{1/2}p^{1/3})\,\Pi_\pm(kp^{-1/3}) \tag{3.13}$$

We note that, by making use of the asymptotic nature of the behaviour of the zeros $-a_n'$ and $-a_n$ as $n \to \infty$ /9/, it is possible to obtain $\Pi_\pm = O(|k|^{-1/4})$ and, therefore, $F_\pm = O(|k|^{1/2})$ as $|k| \to \infty$. By virtue of (3.11) and (3.13), we have

$$\frac{\partial\Phi_-}{\partial y} = -\frac{\mathrm{Ai}'(-a_0^2)}{\mathrm{Ai}(-a_0^2)\,p}\,\frac{k - ia_0'^{1/2}p^{1/3}}{a_0 - a_0'^{1/2}}\,\frac{\Pi_-(kp^{-1/3})}{\Pi_-(ia_0)\,(k - ia_0p^{1/3})} \tag{3.14}$$

In order to prove formula (3.12), let us represent the integral functions $\mathrm{Ai}'(k^2)$ and $\mathrm{Ai}(k^2)$ in the form of an infinite product. By making use of Theorem 2 in Chapter 5 of Sect. 31 in /15/, we obtain (taking account of the numbering of the zeros $-a_n$ and $-a_n'$), after regrouping the factors,

$$\frac{\mathrm{Ai}'(k^2)}{\mathrm{Ai}(k^2)} = \frac{1}{A}\prod_{n=1}^{\infty}\frac{a_n}{a_n'}\,e^{1/(3n)}\,\frac{k^2 + a_0'}{a_0'}\,\exp(-k^2\Sigma_1)\,\Pi_-\Pi_+ \tag{3.15}$$
$$A = \frac{\mathrm{Ai}(0)}{\mathrm{Ai}'(0)}, \quad \Sigma_1 = \frac{1}{A} + \sum_{n=1}^{\infty}\left(\frac{1}{a_n'} - \frac{1}{a_n}\right) + \frac{1}{a_0'}$$

It remains to note that $\Sigma_1 = 0$. In order to prove this equality, let us consider (in the sense of the principal value) the integral

$$\int_{-\infty}^{+\infty}\left(\frac{\mathrm{Ai}'(\lambda^2)}{\lambda\,\mathrm{Ai}(\lambda^2)} - \frac{\lambda\,\mathrm{Ai}(\lambda^2)}{\mathrm{Ai}'(\lambda^2)}\right)d\lambda = 0$$

and close the integration path in the upper half of the λ-plane. We then use the residue theorem.

So, apart from a constant factor, expression (3.15) is identical with (3.12) and the value of this factor can be determined by allowing k to tend to infinity.

Let us now calculate the diffusion flux $\partial c/\partial y$ on the surface of the insert when $z > 0$. We carry out an inverse Fourier transformation of the function $\partial\Phi_-/\partial y$:

$$j' = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\partial\Phi_-}{\partial y}e^{ikz}\,dk \tag{3.16}$$

We substitute (3.14) into (3.16) and close the integration path in the upper half-plane. Then, by using the same reasoning as in the proof of Jordan's lemma /8/, we obtain (by making a_0 tend to zero)

$$j' = -\frac{1}{Ap^{2/3}}\left[1 - \frac{1}{\Pi_-(0)\,a_0'^{1/2}}\sum_{n=1}^{\infty}\frac{a_n^{1/2} - a_0'^{1/2}}{a_n^{1/2}}\Pi^n\exp(-a_n^{1/2}p^{1/3}z)\right] \tag{3.17}$$

$$\Pi^n = -i\lim_{k\to ia_n^{1/2}}\Pi_-(k - ia_n^{1/2})$$

It is also necessary to check postulate (3.6). Assuming that it suffices to verify this when $y = 0$, we obtain from (3.10) and (3.13) that the function Φ_+ is regular when $k = -ia_0p^{1/3}$ but has an infinite number of poles at the points $-ia_n'^{1/2}p^{1/3}$. Closure of the integration contour in the lower half-plane when calculating the inverse Fourier transform shows that, as $z \to -\infty$

$$f = O(\exp\,(a_0'^{1/2} - \varepsilon)p^{1/3}z)$$

where ε is an arbitrary small positive number which confirms (3.6).

Finally, we apply an inverse Laplace transformation to the function j':

$$j(x, z) = \frac{\partial c(x, 0, z)}{\partial y} = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} j'e^{px}\,dp, \quad b > 0 \tag{3.18}$$

By substituting (3.17) into (3.18) and making the change of variables $q = p^{1/3}x^{1/3}$, we obtain (allowing for the definition of the function $\mathrm{Ai}(s)$ in terms of a contour integral /11/)

$$J \equiv jx^{1/3} = -\frac{1}{A\Gamma(2/3)} - \frac{3^{1/3}}{|A|^{1/3}}\left(\frac{2e^{\gamma}}{\pi}\right)^{1/6}\sum_{n=1}^{\infty}\frac{a_n^{1/2} - a_0'^{1/2}}{a_n^{1/2}}\times \tag{3.19}$$

$$\Pi^n\,\mathrm{Ai}\left(\frac{a_n^{1/2}Z}{3^{1/3}}\right)$$

$$Z = zx^{-1/3}$$

In (3.19), use has been made of the equality $\Pi_{\pm}(0)a_0'^{1/2} = (3\pi 2^{-1}e^{-\gamma})^{1/6}/|A|^{1/3}$ which follows from (3.12) when $k = 0$.

The first term in (3.19) is precisely equal to the flow obtained in the boundary layer approximation (the solution of problem (3.1) when $z > 0$ with the term $\partial^2 c/\partial z^2$ omitted), while the second term describes the effect of transverse diffusion (perpendicular to the direction of motion of the flow).

Because the expression for J is rather complex, we shall confine ourselves to finding the principal terms of the asymptotic expansion for J for large and small values of Z.

Let us first consider the case of small Z. After making use of the asymptotic expansion for a_n and a_n' as $n \to \infty$ and the Euler-McLaurin summation formula /10/, we obtain

$$J = \frac{3^{1/6}}{|A|^{1/3}\pi^{1/2}Z^{1/2}}\int_0^{\infty}\frac{\mathrm{Ai}(t)}{t^{1/2}}\,dt + O(1) = \tag{3.20}$$

$$\frac{1}{\Gamma(5/6)(\pi|A|Z)^{1/2}} + O(1) \approx \frac{0.4267}{Z^{1/2}} + O(1)$$

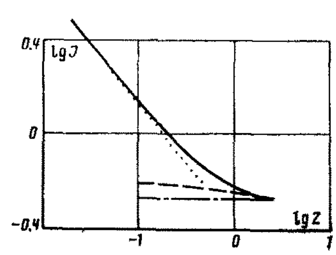

Fig.2

from (3.19). Expression (3.20) confirms the hypothesis regarding the integrability of the singularity as $z \to 0+$.

In the case when $Z \gg 1$, by virtue of the monotonic increase in a_n as n increases and the exponential decay of $\mathrm{Ai}(s)$ for large real and positive values of s, we have

$$J(Z) \sim -\frac{1}{A\Gamma(^2/_3)} - \left(\frac{3}{|A|}\right)^{1/2}\left(\frac{2e^{\gamma}}{\pi}\right)^{1/6}\frac{a_1^{1/2}-a_0^{1/2}}{a_1^{1/2}}\,\Pi^1\,\mathrm{Ai}\left(\frac{a_1^{1/2}Z}{3^{1/3}}\right) \approx \tag{3.21}$$
$$0.5383 + 0.2552\,\mathrm{Ai}\,(1.0602Z), \quad Z \gg 1$$

A log-log plot of $J(Z)$ is shown in Fig.2. The solid line was obtained by numerical summation of the series (3.19), the dotted line represents the first term of the expansion for small values of Z (formula (3.20)), the dot-dash line corresponds to the boundary layer approximation and the broken line to expansion (3.21) for $Z \gg 1$. It is seen that the leading term of expansion (3.20) for small $Z \leqslant 0.1$ (to an accuracy of 4%) is identical with $J(Z)$. When $Z \geqslant 1$ (to an accuracy of 5%) $J(Z)$ is described by expression (3.21). When $Z \geqslant 2$, $J(Z)$ merges with the boundary layer solution, the difference between the two solutions being less than 2%. Hence, the domain of influence of the transverse diffusion when $z > 0$ is approximately bounded by the curve $z \leqslant 2x^{1/3}$.

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